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Braneworlds and dark energy

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Abstract

In the Randall–Sundrum scenario, we analyse the dynamics of an AdS_5 braneworld when conformal matter fields propagate in five dimensions. We show that conformal fields of weight -4 are associated with stable geometries which describe the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic dark energy on a spherically symmetric 3-brane embedded in the compact AdS_5 orbifold. We discuss aspects of the radion stability conditions and of the localization of gravity in the vicinity of the brane.

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1. Introduction

In the Randall–Sundrum (RS) scenario [1, 2], the observable four-dimensional (4D) universe is a 3-brane world embedded in a Z_2 symmetric 5D anti-de Sitter (AdS) space. In the RS1 model [1], the fifth dimension is compact and there are two 3-brane boundaries. The gravitational field is bound to the hidden positive tension brane and decays towards the observable negative tension brane. In this setting, the hierarchy problem is reformulated as an exponential hierarchy between the weak and Planck scales [1]. In the RS2 model [2], the orbifold has an infinite fifth dimension and just one observable positive tension brane near which gravity is exponentially localized.

In the RS models, the classical field dynamics is defined by 5D Einstein equations with a negative bulk cosmological constant Λ_B, Dirac delta sources standing for the branes and a stress–energy tensor describing other fields propagating in the bulk [1–3]. A set of vacuum solutions is given by $d\tilde{s}_5^2 = dy^2 + e^{-2|y|/l}\, ds_4^2$, where y is the Cartesian coordinate representing the fifth dimension, the 4D line element ds_4^2 is Ricci flat, l is the AdS radius given by

$l = 1/\sqrt{-\Lambda_B \kappa_5^2/6}$ with $\kappa_5^2 = 8\pi/M_5^3$ and M_5 is the fundamental 5D Planck mass. In the RS1 model, the hidden Planck brane is located at $y = 0$ and the visible brane at $y' = \pi r_c$, where r_c is the RS compactification scale [1]. The brane tensions $\lambda > 0$ and $\lambda' < 0$ have the same absolute value $|\lambda'| = \lambda$. In the vacuum, λ is given in terms of Λ_B and l by $\lambda = -\Lambda_B l$. In the RS2 model [2], the visible brane is the one with positive tension λ located at $y = 0$. The hidden brane is sent to infinity and is physically decoupled.

The low energy theory of gravity on the observable brane is 4D general relativity, and the cosmology may be Friedmann–Robertson–Walker [1–10]. In the RS1 model, this requires the stabilization of the radion mode using, for example, a 5D scalar field [3, 6, 9, 10]. The gravitational collapse of matter has also been analysed in the RS scenario [11–16]. However, an exact 5D geometry describing a stable black hole localized on a 3-brane has not yet been discovered. Indeed, non-singular localized black holes have only been found in an AdS_4 braneworld [12]. A solution to this problem requires a simultaneous localization of gravity and matter which avoids unphysical divergences [11, 13–16] and could be related to quantum black holes on the brane [15]. In addition, the covariant Gauss–Codazzi approach [17, 18] has uncovered a rich set of braneworld solutions, many of which have not yet been associated with exact 5D spacetimes [19–22].

In this paper we report on research about the dynamics of a spherically symmetric 3-brane in the presence of 5D conformal matter fields [16, 23] (see also [24]). In the previous work [16, 23], we have discovered a new class of exact 5D dynamical solutions for which gravity is bound to the brane by the exponential RS warp. These solutions were shown to be associated with conformal bulk fields characterized by a stress–energy tensor $\tilde{T}_\mu^\nu$ of weight -4 and by the equation of state $\tilde{T}_a^a = 2\tilde{T}_5^5$ (see also [5] and [25]). They were also shown to describe on the brane the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic matter. However, the density and pressures of the conformal bulk fluid increase with the coordinate of the fifth dimension. Consequently and just like in the Schwarzschild black string solution [11], the RS2 scenario is plagued with an unphysical singularity at the AdS horizon. Such divergence does not occur in the RS1 model because the compactified space ends before the AdS horizon is reached. However, the radion mode turns out to be unstable [26]. In this work, we discuss new exact 5D braneworld solutions which are stable under radion field perturbations and still describe on the visible brane the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic dark energy. We also consider the point of view of an effective Gauss–Codazzi observer and show that the gravitational field is bound to the vicinity of the brane.

2. 5D Einstein equations and conformal fields

To map the AdS_5 orbifold, consider the coordinates (t, r, θ, ϕ, z) where z is related to the Cartesian coordinate y by $z = l\,e^{y/l}$, $y > 0$. The most general non-factorizable dynamical metric consistent with the Z_2 symmetry in z and with 4D spherical symmetry on the brane is given by

$$d\tilde{s}_5^2 = \Omega^2 \left(dz^2 - e^{2A}\, dt^2 + e^{2B}\, dr^2 + R^2\, d\Omega_2^2 \right), \tag{1}$$

where $\Omega = \Omega(t, r, z)$, $A = A(t, r, z)$, $B = B(t, r, z)$ and $R = R(t, r, z)$ are Z_2 symmetric functions. $R(t, r, z)$ represents the physical radius of the 2-spheres, and Ω is the warp factor characterizing a global conformal transformation on the metric.

In the RS1 model, the classical dynamics is defined by the 5D Einstein equations,

$$\tilde{G}_\mu^\nu = -\kappa_5^2 \left\{ \Lambda_B \delta_\mu^\nu + \frac{1}{\sqrt{\tilde{g}_{55}}} [\lambda \delta(z - z_0) + \lambda' \delta(z - z'_0)] \left(\delta_\mu^\nu - \delta_5^\nu \delta_\mu^5 \right) - \tilde{T}_\mu^\nu \right\}, \tag{2}$$

where $\tilde{\mathcal{T}}_\mu^\nu$ is the stress–energy tensor of the matter fields which is conserved in 5D,

$$\tilde{\nabla}_\mu \tilde{\mathcal{T}}_\nu^\mu = 0. \tag{3}$$

For a general 5D metric $\tilde{g}_{\mu\nu}$, (2) and (3) form an extremely complex system of differential equations. To solve it, we need simplifying assumptions about the field variables involved in the problem. Let us first consider that the bulk matter is described by conformal fields with weight s. Under the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, the stress–energy tensor satisfies $\tilde{\mathcal{T}}_\mu^\nu = \Omega^{s+2}\mathcal{T}_\mu^\nu$. Consequently, (2) and (3) may be rewritten as [16]

$$G_\mu^\nu = -6\Omega^{-2}(\nabla_\mu\Omega)g^{\nu\rho}\nabla_\rho\Omega + 3\Omega^{-1}g^{\nu\rho}\nabla_\rho\nabla_\mu\Omega - 3\Omega^{-1}\delta_\mu^\nu g^{\rho\sigma}\nabla_\rho\nabla_\sigma\Omega \\ - \kappa_5^2\Omega^2\left\{\Lambda_B\delta_\mu^\nu + \Omega^{-1}[\lambda\delta(z-z_0) + \lambda'\delta(z-z'_0)]\left(\delta_\mu^\nu - \delta_5^\nu\delta_\mu^5\right) - \Omega^{s+2}\mathcal{T}_\mu^\nu\right\}, \tag{4}$$

$$\nabla_\mu\mathcal{T}_\nu^\mu + \Omega^{-1}\left[(s+7)\mathcal{T}_\nu^\mu\partial_\mu\Omega - \mathcal{T}_\mu^\mu\partial_\nu\Omega\right] = 0. \tag{5}$$

If we separate the conformal tensor $\tilde{\mathcal{T}}_\mu^\nu$ into two sectors $\tilde{T}_\mu^\nu$ and $\tilde{U}_\mu^\nu$ with the same weight s, $\tilde{\mathcal{T}}_\mu^\nu = \tilde{T}_\mu^\nu + \tilde{U}_\mu^\nu$ where $\tilde{T}_\mu^\nu = \Omega^{s+2}T_\mu^\nu$ and $\tilde{U}_\mu^\nu = \Omega^{s+2}U_\mu^\nu$, and take $s = -4$ then it is possible to split (4) as follows:

$$G_\mu^\nu = \kappa_5^2 T_\mu^\nu, \tag{6}$$

$$6\Omega^{-2}\nabla_\mu\Omega\nabla_\rho\Omega g^{\rho\nu} - 3\Omega^{-1}\nabla_\mu\nabla_\rho\Omega g^{\rho\nu} + 3\Omega^{-1}\nabla_\rho\nabla_\sigma\Omega g^{\rho\sigma}\delta_\mu^\nu \\ = -\kappa_5^2\Omega^2\left\{\Lambda_B\delta_\mu^\nu + \Omega^{-1}[\lambda\delta(z-z_0) + \lambda'\delta(z-z'_0)]\left(\delta_\mu^\nu - \delta_5^\nu\delta_\mu^5\right)\right\} + \kappa_5^2 U_\mu^\nu. \tag{7}$$

On the other hand, the Bianchi identity implies

$$\nabla_\mu T_\nu^\mu = 0. \tag{8}$$

Then (5) is in fact

$$\nabla_\mu U_\nu^\mu + \Omega^{-1}\left[3\mathcal{T}_\nu^\mu\partial_\mu\Omega - \mathcal{T}_\mu^\mu\partial_\nu\Omega\right] = 0. \tag{9}$$

Note that (6) and (8) are 5D Einstein equations with conformal bulk fields, but without a brane or bulk cosmological constant. They do not depend on the warp factor which is dynamically defined by (7) and (9). The warp is then the only effect reflecting the existence of the brane or of the bulk cosmological constant. We emphasize that this is only possible for the special set of bulk fields which have a stress–energy tensor with conformal weight $s = -4$.

Although the system of dynamical equations is now partially decoupled, it remains difficult to solve. Note for instance that Ω depends nonlinearly on A, B and R. In addition, it is affected by T_μ^ν and U_μ^ν. So consider the special setting $A = A(t,r)$, $B = B(t,r)$, $R = R(t,r)$ and $\Omega = \Omega(z)$. Then (6) and (7) lead to

$$G_a^b = \kappa_5^2 T_a^b, \tag{10}$$

$$G_5^5 = \kappa_5^2 T_5^5, \tag{11}$$

$$6\Omega^{-2}(\partial_z\Omega)^2 + \kappa_5^2\Omega^2\Lambda_B = \kappa_5^2 U_5^5, \tag{12}$$

$$\left\{3\Omega^{-1}\partial_z^2\Omega + \kappa_5^2\Omega^2\{\Lambda_B + \Omega^{-1}[\lambda\delta(z-z_0) + \lambda'\delta(z-z'_0)]\}\right\}\delta_a^b = \kappa_5^2 U_a^b, \tag{13}$$

where the latin indices represent the 4D coordinates t, r, θ and ϕ. Since according to (10) and (11) T_μ^ν depends only on t and r, (8) becomes

$$\nabla_a T_b^a = 0. \tag{14}$$

On the other hand, (12) and (13) imply that U^{ν}_{μ} must be diagonal, $U^{\nu}_{\mu} = \mathrm{diag}(-\bar{\rho}, \bar{p}_{\mathrm{r}}, \bar{p}_{\mathrm{T}}, \bar{p}_{\mathrm{T}}, \bar{p}_5)$, with the density $\bar{\rho}$ and pressures $\bar{p}_{\mathrm{r}}, \bar{p}_{\mathrm{T}}$ satisfying $\bar{\rho} = -\bar{p}_{\mathrm{r}} = -\bar{p}_{\mathrm{T}}$. In addition, U^{ν}_{μ} must only depend on z. Consequently, $\nabla_a U^a_b = 0$ is an identity. Then using (9) and noting that $T^{\nu}_{\mu} = T^{\nu}_{\mu}(t, r)$, we find

$$\partial_z \bar{p}_5 + \Omega^{-1}\partial_z\Omega\left(2U^5_5 - U^a_a\right) = 0, \quad 2T^5_5 = T^a_a. \tag{15}$$

If $U^{\nu}_{\mu}(z)$ is a conserved tensor field like T^{ν}_{μ}, then $\bar{p}_5$ must be constant. So (15) leads to the following equations of state:

$$2T^5_5 = T^a_a, \qquad 2U^5_5 = U^a_a. \tag{16}$$

Then we obtain $\bar{p}_5 = -2\bar{\rho}$. U^{ν}_{μ} is thus constant. On the other hand if $T^{\nu}_{\mu} = \mathrm{diag}\,(-\rho, p_{\mathrm{r}}, p_{\mathrm{T}}, p_{\mathrm{T}}, p_5)$ where ρ, p_{r}, p_{T} and p_5 are, respectively, the density and pressures then its equation of state is rewritten as

$$\rho - p_{\mathrm{r}} - 2p_{\mathrm{T}} + 2p_5 = 0. \tag{17}$$

Note that ρ, p_{r}, p_{T} and p_5 must be independent of z, but may be functions of t and r. The bulk matter is, however, inhomogeneously distributed along the fifth dimension because the physical energy density, $\tilde{\rho}(t, r, z)$, and pressures, $\tilde{p}(t, r, z)$, are related to $\rho(t, r)$ and $p(t, r)$ by the scale factor $\Omega^{-2}(z)$. Also note that T^{ν}_{μ} determines the dynamics on the branes and that in the RS1 model, the two branes have identical cosmological evolutions. On the other hand, it is also important to note that the warp factor depends on the conformal bulk fields only through U^{ν}_{μ}. Consequently, the role of U^{ν}_{μ} is to influence how the gravitational field is warped around the branes. In our previous work U^{ν}_{μ} was set to zero [16, 23, 26]. The corresponding braneworld solutions were warped by the exponential RS scale factor and turned out to be unstable under radion field perturbations [26]. So we also introduce U^{ν}_{μ} as a stabilizing sector.

3. Exact 5D warped solutions

The AdS_5 braneworld dynamics is defined by the solutions of (10) to (14) and (17). Let us first solve (12) and (13). As we have seen, U^{ν}_{μ} is constant with $\bar{\rho} = -\bar{p}_{\mathrm{r}} = -\bar{p}_{\mathrm{T}} = -\bar{p}_5/2$. If $\bar{p}_5 = 0$ then $U^{\nu}_{\mu} = 0$, and we end up with the usual RS warp equations. As is well known, a solution is the exponential RS warp $\Omega(y) = \Omega_{\mathrm{RS}}(y) = \mathrm{e}^{-|y|/l}$ [1, 2]. If $\bar{p}_5$ is non-zero then we find a new set of warp solutions. Integrating (12) and taking into account the Z_2 symmetry, we obtain (see figure 1)

$$\Omega(y) = \mathrm{e}^{-|y|/l}\left(1 + p^5_{\mathrm{B}}\,\mathrm{e}^{2|y|/l}\right), \tag{18}$$

where $p^5_{\mathrm{B}} = \bar{p}_5/(4\Lambda_{\mathrm{B}})$. This set of solutions must also satisfy (13) which contains the Israel jump conditions. As a consequence, the brane tensions λ and λ' are given by

$$\lambda = \lambda_{\mathrm{RS}}\frac{1 - p^5_{\mathrm{B}}}{1 + p^5_{\mathrm{B}}}, \qquad \lambda' = -\lambda_{\mathrm{RS}}\frac{1 - p^5_{\mathrm{B}}\exp(2\pi r_{\mathrm{c}}/l)}{1 + p^5_{\mathrm{B}}\exp(2\pi r_{\mathrm{c}}/l)}, \tag{19}$$

where $\lambda_{\mathrm{RS}} = 6\big/\left(l\kappa^2_5\right)$. Note that in the limit $p^5_{\mathrm{B}} \to 0$, we obtain the RS warp and also the corresponding brane tensions.

To determine the dynamics on the brane, we need to solve (10) and (11) when T^{ν}_{μ} satisfies (14) and (17). Note that as long as p_5 balances ρ, p_{r} and p_{T} according to (11) and (17), the 4D equation of state is not constrained. Three examples corresponding to inhomogeneous dust, generalized dark radiation and homogeneous polytropic matter were analysed in [16] and [23]. The latter describes the dynamics on the brane of dark energy in the form of a polytropic fluid. The diagonal conformal matter may be defined by

$$\rho = \rho_{\mathrm{P}}, \qquad p_{\mathrm{r}} + \eta\rho_{\mathrm{P}}{}^{\alpha} = 0, \qquad p_{\mathrm{T}} = p_{\mathrm{r}}, \qquad p_5 = -\tfrac{1}{2}\left(\rho_{\mathrm{P}} + 3\eta\rho_{\mathrm{P}}{}^{\alpha}\right), \tag{20}$$

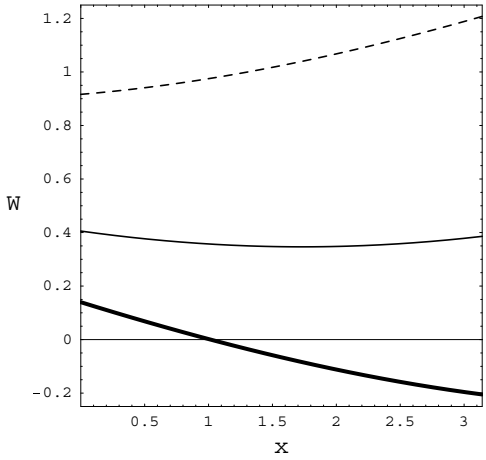

Figure 1. Plots of $W = \ln \Omega(x), x = y/r_c$ for $l/r_c = 5$. The dashed, thin and thick lines correspond, respectively, to p_B^5 equal to 1.5, 0.5 and 0.15.

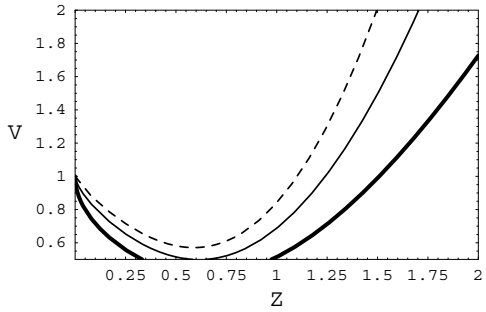

Figure 2. Plots of $V = S\dot{S}^2, Z = S^{1-\alpha}$ for $k > 0, \eta > 0$ and $a > 0$. The dashed, thin and thick lines correspond, respectively, to α equal to $-1/4, -1/2$ and -1.

where ρ_{P} is the polytropic energy density and the parameters (α, η) characterize different polytropic phases.

Solving the conservation equations, we find [23, 27]

$$\rho_{\mathrm{P}} = \left(\eta + \frac{a}{S^{3-3\alpha}}\right)^{\frac{1}{1-\alpha}}, \tag{21}$$

where $\alpha \neq 1$, a is an integration constant and $S = S(t)$ is the Robertson–Walker scale factor of the brane world which is related to the physical radius by $R = rS$. For $-1 \leqslant \alpha < 0$, the fluid is in its generalized Chaplygin phase (see also [27]). With this density, the Einstein equations lead to the following 5D dark energy polytropic solutions [23]:

$$\mathrm{d}\tilde{s}_5^2 = \Omega^2 \left[-\mathrm{d}t^2 + S^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega_2^2\right)\right] + \mathrm{d}y^2, \tag{22}$$

where the brane scale factor S satisfies $\dot{S}^2 = \kappa_5^2 \rho_{\mathrm{P}} S^2/3 - k$. The global evolution of the observable universe is then given by [22, 23]

$$S\dot{S}^2 = V(S) = \frac{\kappa_5^2}{3}\left(\eta S^{3-3\alpha} + a\right)^{\frac{1}{1-\alpha}} - kS. \tag{23}$$

In figure 2, we present some illustrative examples.

4. Radion stability

To analyse how these solutions behave under radion field perturbations, we consider the saddle point expansion of the RS action [26, 28, 29]

$$\tilde{S} = \int \mathrm{d}^4x\,\mathrm{d}y\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2\kappa_5^2} - \Lambda_\mathrm{B} - \frac{1}{\sqrt{\tilde{g}_{55}}}[\lambda\delta(y) + \lambda'\delta(y-\pi r_\mathrm{c})] + \tilde{L}_\mathrm{B}\right\}, \tag{24}$$

where $\tilde{L}_\mathrm{B}$ is the Lagrangian characterizing the 5D matter fields. The most general metric consistent with the Z_2 symmetry in y and with 4D spherical symmetry on the brane may be written in the form

$$\mathrm{d}\tilde{s}^2 = a^2\,\mathrm{d}s_4^2 + b^2\,\mathrm{d}y^2, \qquad \mathrm{d}s_4^2 = -\mathrm{d}t^2 + \mathrm{e}^{2B}\,\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega_2^2, \tag{25}$$

where the metric functions $a = a(t, r, y)$, $B = B(t, r, y)$, $R = R(t, r, y)$ and $b = b(t, r, y)$ are Z_2 symmetric. Now a is the warp factor and b is related to the radion field. Our braneworld backgrounds correspond to $b = 1$, $B = B(t, r)$, $R = R(t, r)$ and $a = \Omega(y)$.

Consider (25) with $a(t, r, y) = \Omega(y)\,\mathrm{e}^{-\beta(t,r)}$ and $b(t, r) = \mathrm{e}^{\beta(t,r)}$. Then the dimensional reduction of (24) in the Einstein frame leads to [26]

$$\tilde{S} = \int \mathrm{d}^4x\sqrt{-g_4}\left(\frac{R_4}{2\kappa_4^2} - \frac{1}{2}\nabla_c\gamma\nabla_d\gamma\, g_4^{cd} - \tilde{V}\right), \tag{26}$$

where $\gamma = \beta/(\kappa_4\sqrt{2/3})$ is the canonically normalized radion field. The function $\tilde{V} = \tilde{V}(\gamma)$ is the radion potential, and it is given by

$$\tilde{V} = \frac{2}{\kappa_5^2}\chi^3\int \mathrm{d}y\Omega^2\left[3(\partial_y\Omega)^2 + 2\Omega\partial_y^2\Omega\right] + \chi\int \mathrm{d}y\Omega^4(\Lambda_\mathrm{B} - \tilde{L}_\mathrm{B}) + \chi^2\int \mathrm{d}y\Omega^4[\lambda\delta(y) + \lambda'\delta(y-\pi r_\mathrm{c})], \tag{27}$$

where $\chi = \exp(-\kappa_4\gamma\sqrt{2/3})$, and we have chosen $\int_{-\pi r_\mathrm{c}}^{\pi r_\mathrm{c}} \mathrm{d}y\Omega^2 = \kappa_5^2/\kappa_4^2$.

To analyse the stability of the AdS_5 braneworld solutions, we consider the saddle point expansion of the radion field potential $\tilde{V}$. If $p_\mathrm{B}^5 = 0$, then $\Omega = \Omega_\mathrm{RS}$. The radion potential has two critical extrema, $\chi_1 = 1$ and $\chi_2 = 1/3$ [26]. Our solutions correspond to the first root $\chi_1 = 1$. The same happens if the bulk matter is absent as in the RS vacuum solutions. Stable background solutions must be associated with a positive second variation of the radion potential. If the equation of state of the conformal bulk fields is independent of the radion perturbation, then for $\chi = \chi_1 = 1$ the second variation is negative, and so the braneworld solutions are unstable [26].

If the equation of state is kept invariant under the radion perturbations, it is possible to find stable solutions at $\chi = 1$ if the warp is changed. Indeed, the new relevant warp functions are given in (18). Consider $\tilde{\mathcal{V}} = \int \mathrm{d}^4x\sqrt{-g_4}\tilde{V}$. With $x = y/r_\mathrm{c}$ and $r_\mathrm{c}\int_{-\pi}^{\pi}\mathrm{d}x\Omega^2 = \kappa_5^2/\kappa_4^2$, we find

$$\left.\frac{\delta^2\tilde{\mathcal{V}}}{\delta\gamma^2}\right|_{\gamma=0} = -\frac{4}{3}\kappa_4^2\left(r_\mathrm{c}^2\int \mathrm{d}x\Omega^2\right)^{-1}\int \mathrm{d}^4x\sqrt{-\tilde{g}_4}M, \tag{28}$$

where the dimensionless radion mass parameter M is

$$M = \lambda r_\mathrm{c}\kappa_5^2\Omega^4(0) + \lambda' r_\mathrm{c}\kappa_5^2\Omega^4(\pi) - \frac{6r_\mathrm{c}^2}{l^2}\int \mathrm{d}x\Omega^4. \tag{29}$$

Stable solutions correspond to $M < 0$. Consequently, stability exists for a range of the model parameters if $p_\mathrm{B}^5 > 0$ (see figure 3). For $p_\mathrm{B}^5 \leqslant 0$, all solutions are unstable. For $p_\mathrm{B}^5 > 0$, the stability of the AdS_5 braneworlds also depends on the dimensionless ratio l/r_c.

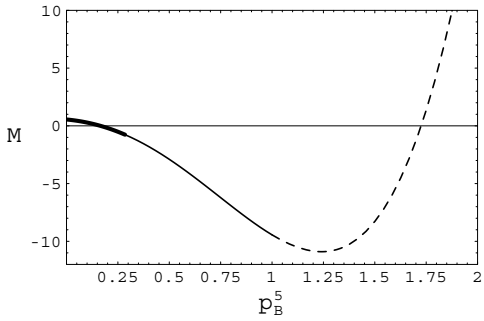

Figure 3. Plot of radion mass parameter M for $l/r_c = 5$. Thick line, $0 < p_B^5 \leqslant e^{-2\pi/5} : \lambda > 0$, $\lambda' \leqslant 0$. Thin line, $e^{-2\pi/5} < p_B^5 \leqslant 1 : \lambda \geqslant 0, \lambda' > 0$. Dashed line, $p_B^5 > 1 : \lambda < 0, \lambda' > 0$.

For $l/r_c < 1.589\ldots$, all solutions turn out to be unstable. Stable universes begin to appear at $l/r_c = 1.589\ldots$, $p_B^5 = 0.138\ldots$. For $l/r_c > 1.589\ldots$, we find stable solutions for an interval of p_B^5 (see the example of $l/r_c = 5$ in figure 3), which increases with l/r_c. For large enough but finite l/r_c, the stability interval approaches the limit $]0.267\ldots, 3.731\ldots[$. Naturally, $M \to 0$ if $l/r_c \to \infty$.

5. Gauss–Codazzi equations and localization of gravity

For an observer confined to the brane, the effective 4D Einstein equations are given by [16–18, 21]

$$\mathcal{G}_\mu^\nu = \frac{2\kappa_5^2}{3}\left[\mathcal{U}_\alpha^\beta q_\mu^\alpha q_\beta^\nu + \left(\mathcal{U}_\alpha^\beta n^\alpha n_\beta - \frac{1}{4}\mathcal{U}_\alpha^\alpha\right) q_\mu^\nu\right] + \mathcal{K}_\alpha^\alpha \mathcal{K}_\mu^\nu - \mathcal{K}_\mu^\alpha \mathcal{K}_\alpha^\nu - \frac{1}{2} q_\mu^\nu \left(\mathcal{K}^2 - \mathcal{K}_\alpha^\beta \mathcal{K}_\beta^\alpha\right) - \mathcal{E}_\mu^\nu, \tag{30}$$

where $\mathcal{G}_\mu^\nu = G_\alpha^\beta q_\mu^\alpha q_\beta^\nu$, $n^\mu = \delta_5^\mu$ is the unit normal to the brane and $q_\mu^\nu = \delta_\mu^\nu - n_\mu n^\nu$. The stress–energy tensor is $\mathcal{U}_\mu^\nu = -\Lambda_B \Omega^2(0)\delta_\mu^\nu + \mathcal{T}_\mu^\nu$, $\mathcal{K}_\mu^\nu$ is the extrinsic curvature and $\mathcal{E}_\mu^\nu$ is the traceless projection of the 5D Weyl tensor. The 4D observer finds the same dynamics on the brane because [16]

$$\mathcal{E}_a^b = \frac{\kappa_5^2}{3}\left(-T_a^b + \frac{1}{2}T_5^5 \delta_a^b\right) \tag{31}$$

and

$$\frac{4}{3}\left(U_a^b + \frac{1}{4}U_5^5 \delta_a^b\right) - \left(\Lambda_B + \frac{\kappa_5^2 \lambda^2}{6}\right)\delta_a^b \Omega^2(0) = 0. \tag{32}$$

Since the tidal acceleration [16, 21] is $a_T = \kappa_5^2 \Lambda_B \left(1 + p_B^5\right)^2 / 6 < 0$, the gravitational field is bound to the vicinity of the brane.

6. Conclusions

In this paper we have analysed exact 5D solutions describing the dynamics of AdS_5 braneworlds when conformal fields of weight -4 are present in the bulk. We have discussed their behaviour under radion field perturbations and shown that if the equation of state characterizing the

conformal fluid is independent of the perturbation, then the radion may be stabilized by a sector of the conformal fields while another sector of the same class of fields generates the dynamics on the brane. Stabilization requires a bulk fluid sector with a constant negative 5D pressure and involves new warp functions. On the brane these solutions are able to describe, for example, the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic dark energy. More general 4D equations of state may also be considered. This analysis is left for future work. We have also shown that an effective Gauss–Codazzi observer sees gravity localized near the brane and deduces the same dynamics on the brane if she makes the same hypothesis about the 5D fields. Whether gravity is sufficiently bound to the brane and the hierarchy is strong enough are open problems for future research.

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